

Dynamics of a spinning particle in an EM field

Relativistic treatment of angular momentum

$J^{\mu\nu}$ second rank antisymmetric tensor

$$J^i = \frac{1}{2} \epsilon^{ijk} J_{jk}, \quad K^i = J^{0i}$$

J^i = components of the angular momentum

K^i = related to Lorentz boosts

(associated with the motion of the center of mass
of a system)

Decompose:

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$$

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$$

$$\Rightarrow L^i = \frac{1}{2} \epsilon^{ijk} L_{jk}$$

$$\vec{L} = \vec{x} \times \vec{p} \quad ("orbital" \text{ angular momentum})$$

$$\text{Likewise } S^i = \frac{1}{2} \epsilon^{ijk} S_{jk} \quad ("spin" \text{ angular momentum})$$

Pauli-Lubanski

$$w^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\lambda} J_{\nu\rho} P_\lambda \quad (\epsilon^{0123} = 1)$$

$$w^\mu = (\vec{J} \cdot \vec{P}; P^0 \vec{J} + \vec{K} \times \vec{P})$$

$$\begin{aligned}\vec{J} &= \vec{L} + \vec{S} \\ &= \vec{x} \times \vec{P} + \vec{S}\end{aligned}$$

Note

$$w \cdot P = 0$$

$$\text{since } \epsilon^{\mu\nu\lambda} P_\lambda P_\mu = 0$$

$$\text{Go to the rest frame: } P^\mu = (mc; \vec{0}) \quad (u \neq 0)$$

$$\Rightarrow w^\mu = (0; mc \vec{S})$$

$$w^2 \equiv w \cdot w = -m^2 c^2 \vec{S}^2$$

This is an intrinsic property of the particle

It is convenient to normalize w^μ

$$S^\mu = \frac{w^\mu}{(-w^2)^{1/2}}$$

Key properties

$$S \cdot P = 0, \quad S \cdot S = -1$$

$$\text{In the rest frame, } S^\mu = (0; \vec{S})$$

\vec{S} points in the direction of the spin

Dynamics of spin

particle in motion (may be accelerating)

In the laboratory frame, how does \vec{s} evolve.

In the lab frame $\vec{v}(t) = c\vec{\beta}(t)$ K

In the instantaneous rest frame K'

$$S'^\mu = (0; \vec{s}) \quad \frac{dS'^\mu}{dt} = \left(\frac{dS'^0}{dt}; \vec{0} \right)$$

Why $\frac{dS'^0}{dt} \neq 0$? if there is no external torque

Because K' at time $T+dt$
is a different inertial reference frame
compared to K' at time T .

Start with $S \cdot u = 0$ $\overset{u}{p} = m u^\mu$

$$u \cdot \frac{dS}{dt} + S \cdot \frac{du}{dt} = 0$$

$$\frac{du^\mu}{dt} = \frac{\gamma^3}{c^2} \left(\vec{v} \cdot \frac{d\vec{v}}{dt} \right) u^\mu + \left(0; \gamma^2 \frac{d\vec{v}}{dt} \right)$$

$$\gamma = \left(1 - \frac{v^2}{c^2} \right)^{-1/2}$$

In the rest frame, $u'^\mu = (c; \vec{0})$

$$\begin{array}{l} \vec{v}' = 0 \\ \gamma = 1 \end{array}$$

$$\frac{dS'^0}{dT} = \gamma \cdot \frac{d\vec{\beta}'}{dT} \quad \vec{\beta}' = 0$$



$$\frac{dS'^\mu}{dT} = K u'^\mu \quad K = \text{Lorentz invariant scalar}$$

\Rightarrow true in all ^{inertial} reference frames.

$$\frac{dS^\mu}{dT} = K u^\mu$$

$$u \cdot \frac{dS}{dT} = K u \cdot u = K c^2$$

$$\Rightarrow S \cdot \frac{du}{dT} = -K c^2$$

$$K = -\frac{1}{c^2} S \cdot \frac{du}{dT}$$

$$\boxed{\frac{dS^\mu}{dT} = -\frac{1}{c^2} \left(S \cdot \frac{du}{dT} \right) u^\mu}$$

Covariant equation for S^μ

Workout $S \cdot \frac{du}{d\tau}$ in the lab frame

$$S \cdot \frac{du}{d\tau} = -\gamma^2 c \vec{S} \cdot \frac{d\vec{\beta}}{dt} \quad (c\vec{\beta} = \vec{v})$$

$$\frac{dS^u}{d\tau} = \frac{\gamma^2}{c} \left(\vec{S} \cdot \frac{d\vec{\beta}}{dt} \right) u^u$$

In lab frame, $d\tau = \gamma^{-1} dt$

$$\frac{dS^o}{dt} = \gamma^2 \vec{S} \cdot \frac{d\vec{\beta}}{dt}$$

$$\frac{d\vec{S}}{dt} = \gamma^2 \left(\vec{S} \cdot \frac{d\vec{\beta}}{dt} \right) \vec{\beta}$$

But how does \vec{S} evolve with t

Boost back from lab to the instantaneous rest frame

using $\Lambda = \begin{pmatrix} \gamma & -\gamma \vec{\beta} \\ -\gamma \vec{\beta}^\top & \delta_{ij} + (\gamma-1) \frac{\beta_i \beta_j}{\beta^2} \end{pmatrix}$

$$\vec{S}' = \hat{\vec{S}} = \vec{S} + \frac{\gamma^2}{\gamma+1} (\vec{\beta} \cdot \vec{S}) \vec{\beta} - \gamma \vec{\beta} \vec{S}$$

$$S'^0 = 0 = \gamma(S^0 - \vec{\beta} \cdot \vec{S})$$

$$\Rightarrow S^0 = \vec{\beta} \cdot \vec{S}$$

$$\Rightarrow \hat{\vec{S}} = \vec{S} - \frac{\gamma}{\gamma+1} \vec{\beta} (\vec{\beta} \cdot \vec{S})$$

$$\begin{aligned} \frac{d\hat{\vec{S}}}{dt} &= \frac{d\vec{S}}{dt} - \vec{\beta}(\vec{\beta} \cdot \vec{S}) \frac{d}{dt} \left(\frac{\gamma}{\gamma+1} \right) \\ &\quad - \frac{\gamma}{\gamma+1} \left\{ \vec{\beta} \left(\vec{\beta} \cdot \frac{d\vec{S}}{dt} + \vec{S} \cdot \frac{d\vec{\beta}}{dt} \right) + (\vec{\beta} \cdot \vec{S}) \frac{d\vec{\beta}}{dt} \right\} \end{aligned}$$

End result of a page of algebra

$$\begin{aligned} \frac{d\hat{\vec{S}}}{dt} &= \frac{\gamma^2}{\gamma+1} \left[\left(\vec{S} \cdot \frac{d\vec{\beta}}{dt} \right) \vec{\beta} - (\vec{\beta} \cdot \vec{S}) \frac{d\vec{\beta}}{dt} \right] \\ &= \frac{\gamma^2}{\gamma+1} \vec{S} \times \left(\vec{\beta} \times \frac{d\vec{\beta}}{dt} \right) \quad \vec{a} = \frac{d\vec{v}}{dt} \end{aligned}$$

$$\frac{d\vec{S}}{dt} = \vec{\omega}_T \times \vec{S}, \quad \boxed{\vec{\omega}_T = \frac{\gamma^2}{\gamma+1} \frac{\vec{a} \times \vec{v}}{c^2}}$$

Thomas precession.

Introduce an external EM field

A particle with spin is also a magnetic moment

$$\vec{m}' = \frac{ge}{2mc} \vec{s}' \quad e = \text{charge of the particle}$$

Torque $\vec{N}' = \vec{m}' \times \vec{B}'$

$$\frac{d\vec{s}}{dt} = \frac{ge}{2mc} \vec{s} \times \vec{B}'$$

$$\frac{dS'^\mu}{dt} = \left(\vec{s} \cdot \frac{d\vec{p}'}{dt}; \frac{ge}{2mc} \vec{s} \times \vec{B}' \right)$$

$$= \frac{ge}{2mc} F'^{\mu\nu} S'_\nu + \left(\vec{s} \cdot \frac{d\vec{p}'}{dt} - \frac{ge}{2mc} \vec{s} \cdot \vec{E}'; \vec{0} \right)$$

$$= \frac{ge}{2mc} F'^{\mu\nu} S'_\nu + \tilde{K} u^\mu$$



$$\frac{dS^\mu}{dt} = \frac{ge}{2mc} F^{\mu\nu} S_\nu + \tilde{K} u^\mu$$

$$\Rightarrow \tilde{K} = -\frac{1}{c^2} S \cdot \frac{du}{dt} + \frac{qe}{2mc^3} S_\alpha F^{\alpha\beta} u_\beta$$

Conclude

$$\boxed{\frac{dS^\mu}{dt} = \frac{qe}{2mc} \left[F^{\mu\nu} S_\nu + \frac{1}{c^2} u^\mu S_\alpha F^{\alpha\beta} u_\beta \right] - \frac{1}{c^2} \left(S \cdot \frac{du}{dt} \right) u^\mu}$$

eq (11.162) of Jackson

Using the Lorentz force equation

$$\frac{du^\alpha}{dt} = \frac{e}{mc} F^{\alpha\beta} u_\beta$$

$$\vec{F} = e \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right)$$

$$\boxed{\frac{dS^\mu}{dt} = \frac{e}{mc} \left[\frac{q}{2} F^{\mu\nu} S_\nu + \frac{1}{c^2} \left(\frac{q}{2} - 1 \right) u^\mu S_\alpha F^{\alpha\beta} u_\beta \right]}$$

BMT equation

secret assumption: gradient forces are neglected

$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$$

Finally, we want $\frac{d\vec{s}}{dt}$. After two pages of algebra,

$$\boxed{\frac{d\vec{s}}{dt} = \frac{e}{mc} \vec{s} \times \left\{ \left(\frac{g}{2} - 1 + \frac{1}{\gamma} \right) \vec{B} - \left(\frac{g}{2} - 1 \right) \frac{\gamma}{\gamma+1} (\vec{\beta} \cdot \vec{B}) \vec{\beta} - \left(\frac{g}{2} - \frac{\gamma}{\gamma+1} \right) \vec{\beta} \times \vec{E} \right\}}$$

Thomas equation

eq. (11.170)

of Jackson

$$\frac{d\vec{\beta}}{dt} = \frac{e}{\gamma mc} [\vec{E} + \vec{\beta} \times \vec{B} - \vec{\beta}(\vec{\beta} \cdot \vec{E})]$$

(Remark: Thomas precession is regained if $g=0$)

Application: $\vec{E}=0$, $\vec{\beta} \cdot \vec{B}=0$

$$\frac{d\vec{s}}{dt} = \frac{e}{\gamma mc} \left[1 + \frac{1}{2}(g-2)\gamma \right] \vec{s} \times \vec{B}$$

$$\frac{d\vec{v}}{dt} = \vec{v} \times \vec{\omega}_B \quad \vec{\omega}_B = \frac{e \vec{B}}{\gamma mc}$$

g -factor of e^- , μ^- is $g \approx 2$

Application: spin-orbit Hamiltonian in atomic physics

A famous factor of 2 discrepancy is resolved by including Thomas precession.

Energy-momentum tensor of electrodynamics

$$\Theta^{\mu\nu} = \frac{1}{4\pi} \left[\frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\alpha} F^\nu_\alpha \right]$$

$$\Theta^{ij} = -\frac{1}{4\pi} [E^i E^j + B^i B^j - \frac{1}{2} \delta^{ij} (\vec{E}^2 + \vec{B}^2)]$$

$= -T^{ij}$ Maxwell stress tensor

$$\Theta^{00} = \frac{1}{8\pi} (\vec{E}^2 + \vec{B}^2)$$

$$\Theta^{i0} = \Theta^{0i} = \frac{1}{4\pi} (\vec{E} \times \vec{B})^i$$

properties of $\Theta^{\mu\nu}$ (traceless symmetric 2nd rank tensor)

(1) $\Theta^{\mu\nu} = \Theta^{\nu\mu}$; (2) $\Theta^\mu_\mu = 0$ (sum over μ)

$$\Theta^{\mu\nu} = \begin{pmatrix} u & | & \frac{\vec{S}}{c} \equiv c\vec{g} \\ - & - & - - - - \\ \vec{c}\vec{g} & | & -T^{ij} \end{pmatrix}$$

u = energy density

$\frac{\vec{S}}{c}$ = (energy flux) / c
or energy current density

\vec{S} = Poynting vector

$c\vec{g}$ = c (momentum density)

$-T^{ij}$ = stress tensor
or
momentum current density

We have distinguished the physical significance of Θ^{0i} and Θ^{i0}
although $\Theta^{0i} = \Theta^{i0}$.

In the absence of sources, energy-momentum is conserved.

$$\partial_\mu \Theta^{\mu\nu} = 0$$

$$P^\nu = \frac{1}{c} \int \Theta^{0\nu} d^3x$$

For $\nu=0$, energy conservation

For $\nu=i=1, 2, 3$ momentum conservation

Recall that

$$\partial_\mu S^\mu = 0 \Rightarrow \vec{\nabla} \cdot \vec{S} + \frac{1}{c} \frac{\partial S^0}{\partial t} = 0$$

$$Q = \int S^0 d^3x$$

$$\frac{dQ}{dt} = \int \frac{\partial S^0}{\partial t} d^3x = -c \int \vec{\nabla} \cdot \vec{S} d^3x = 0$$

Hence

$$\boxed{\frac{dP^\nu}{dt} = 0}$$

In the presence of sources

$$\partial_\mu \Theta^{\mu\nu} = \frac{1}{c} F^{\mu\nu} J_\mu$$

This is the relativistic generalization of
Poynting's theorem (see Chapter 6 of
Jackson)

Angular momentum tensor

$$M^{\alpha\beta\gamma} = \Theta^{\alpha\gamma} x^\beta - \Theta^{\beta\gamma} x^\alpha$$

$$\partial_\alpha M^{\alpha\beta\gamma} = 0$$

Proof: use $\partial_\mu \Theta^{\mu\nu} = 0$, $\Theta^{\mu\nu} = \Theta^{\nu\mu}$

$$J^{\mu\nu} = \frac{1}{c} \int M^{0\mu\nu} d^3x$$

$$\frac{d J^{\mu\nu}}{dt} = 0 \quad \text{in the absence of sources}$$